Pergamon

J. Appl. Maths Mechs, Vol. 57, No. 6, pp. 965–972, 1993

0021–8928/93 \$24.00+0.00

0021–8928(94)E0002-R

STABILITY OF DISTRIBUTED-PARAMETER SYSTEMS WITH RETARDED ARGUMENT†

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(*Received* 11 *November* 1992)

Lyapunov functions are used to investigate the stability of processes described by a system of linear partial differential equations with retarded argument (for example: magnetohydrodynamic processes, elastic vibrations in aircraft, etc.). Some equations of the system may not involve time derivatives (for example, the equation of continuity in incompressible fluid flow, and the equation for the magnetic induction vector in the theory of electromagnetic phenomena). Such equations also arise when the order of a partial differential equation is reduced by introducing new notation for the space derivatives. A method is developed for investigating the stability of processes described by a system of this kind, some of whose equations do not contain time derivatives. Two constructions of the Lyapunov functions, as different integral quadratic forms, are proposed. Sufficient conditions for stability, in the form of inequalities relating the coefficients of the system, are established. As an example, the stability of the vibrations of a stretched string in a viscoelastic medium due to a distributed control force is considered.

1. STATEMENT OF THE PROBLEM

CONSIDER a perturbed process with distributed parameters, described by the following system of partial differential equations with a retarded argument

$$\frac{\partial\varphi}{\partial t}=A(x)\frac{\partial\varphi}{\partial x}+B(x)\frac{\partial\psi}{\partial x}+A_0(x)\varphi+B_0(x)\psi+Q(x)\varphi_\tau \tag{1.1}$$

$$C(x)\frac{\partial\varphi}{\partial x}+D(x)\frac{\partial\psi}{\partial x}+C_0(x)\varphi+D_0(x)\psi=0 \tag{1.2}$$

$$t\in[t_0,\infty),\quad x\in(0,1),\quad \varphi=\varphi(x,t),\quad \varphi_\tau=\varphi(x,t-\tau)$$

where φ and φ_τ are n-vectors of phase functions, $\psi=\psi(x,t)$ is the m-vector of phase functions whose time derivative does not occur in system (1.1), (1.2), $A(x)$, $A_0(x)$, $B(x)$, $B_0(x)$, $C(x)$, $C_0(x)$, $D(x)$, $D_0(x)$ and $Q(x)$ are matrices whose elements are absolutely continuous functions, τ is the delay of the argument t.

We assume that the initial values of the vector-valued function $\varphi(x,t)$ lie in the space $L_2([0,1]\times[t_0-\tau,t_0])$; homogeneous boundary conditions are imposed at the endpoints $x=0$, $x=1$ of the interval $(0,1)$

$$\alpha_0\varphi(0,t)+\beta_0\psi(0,t)=0,\quad \alpha_1\varphi(1,t)+\beta_1\psi(1,t)=0 \tag{1.3}$$

†*Prikl. Mat. Mekh.* Vol. 57, No. 6, pp. 14–21, 1993.

where $\alpha_i\beta_i$ $(i=0, 1)$ are matrices whose elements are continuous bounded functions of time.

We wish to solve system (1.1)–(1.3) in the class of absolutely continuous functions of x.

We define a measure of the deviation of the perturbed process from the unperturbed process $\varphi\equiv\psi\equiv 0$ at an arbitrary time $t\geqslant t_0$

$$\rho[\varphi(\cdot,t)]=\int_0^1 \varphi^T(x,t)\varphi(x,t)dx \tag{1.4}$$

and a measure $\rho_0[\varphi]=0$, constraining the initial perturbations.

Definition 1. The measure $\rho[\varphi]$ is said to be continuous at time $t=t_0$ with respect to the measure $\rho_0[\varphi]$ at $\rho_0[\varphi]=0$, if, for any number $\varepsilon>0$, a number $\delta=\delta(\varepsilon)>0$ exists such that $\rho[\varphi]<\varepsilon$ whenever $\rho_0[\varphi]<\delta$ and $t\geqslant t_0$.

Henceforth we shall assume that $\rho[\varphi]$ is continuous with respect to $\rho_0[\varphi]$ at $t=t_0$ and $\rho_0[\varphi]=0$. For example, this condition holds for the following initial measures

$$\rho_{01}[\varphi]=\sup_{s\in[t_0-\tau,t_0]}\rho[\varphi(\cdot,s]\tag{1.5}$$

$$\rho_{02}[\varphi]=\rho[\varphi(\cdot,t_0)]+\int_{t_0-\tau}^{t_0}\rho[\varphi(\cdot,s]ds\tag{1.6}$$

which we will indeed use in examining certain specific problems.

Definition 2. The unperturbed process $\varphi\equiv\psi\equiv 0$ is said to be stable with respect to the two measures $\rho[\varphi]$ and $\rho_0[\varphi]$ if, for any preassigned number $\varepsilon>0$, one can find a number $\delta=\delta(\varepsilon)>0$ such that, for all admissible initial distributions satisfying the condition $\rho_0[\varphi]<\delta$, it is true at any time $t\geqslant t_0$ that $\rho[\varphi]<\varepsilon$.

Modification of the method of Lyapunov functions will yield sufficient conditions for a solution $\varphi\equiv\psi\equiv 0$ of system (1.1)–(1.3) to be stable with respect to $\rho[\varphi]$ and $\rho_0[\varphi]$. For the case of no delay, i.e. $Q(x)\equiv 0$, sufficient conditions for the trivial solution of system (1.1)–(1.3) to be stable with respect to $\rho[\varphi]$ have already been established [1].

The special feature of this system is that Eq. (1.2) does not involve derivatives with respect to t. This means that one cannot directly evaluate the derivative of the Lyapunov function with respect to time along trajectories of the process described by the system. We will propose a procedure somewhat similar to the method of Lagrange multipliers in variational problems. The method will be described using various forms of Lyapunov functions.

2. FIRST METHOD OF CONSTRUCTING LYAPUNOV FUNCTIONS

Here we will use an idea developed in [2] to investigate the stability of systems with lumped parameters and delay, which was extended in [3] to delay systems with distributed parameters.

To solve the problem, we propose to use the function

$$V[\varphi(\cdot,t)]=\int_0^1 \varphi^T(x,t)\upsilon(x)\varphi(x,t)dx\tag{2.1}$$

where $\upsilon(x)$ is a matrix whose elements are absolutely continuous functions. The derivative dV/dt along trajectories of Eq. (1.1) is

$$\frac{dV}{dt}=\int_0^1\left\{\varphi^T(\upsilon A_0+A_0^T\upsilon)\varphi+\varphi^T\upsilon A\frac{\partial\varphi}{\partial x}+\frac{\partial\varphi^T}{\partial x}A^T\upsilon\varphi+\right.$$
$$\left.+\varphi^T\upsilon B\frac{\partial\psi}{\partial x}+\frac{\partial\psi^T}{\partial x}B^T\upsilon\varphi+\varphi^T\upsilon B_0\psi+\psi^T B_0^T\upsilon\varphi+2\varphi^T\upsilon Q\varphi_\tau\right\}dx\tag{2.2}$$

To take Eq. (1.2) into consideration, we add the following equality to expression (2.2)

$$\int_0^1 \left\{(\varphi^T P_1 + \psi^T P_2)\left[C\frac{\partial\varphi}{\partial x} + D\frac{\partial\psi}{\partial x} + C_0\varphi + D_0\psi\right] + \right.$$

$$\left. + \left[\frac{\partial\varphi^T}{\partial x}C^T + \frac{\partial\psi^T}{\partial x}D^T + \varphi^T C_0^T + \psi^T D_0^T\right](P_1^T\varphi + P_2^T\psi)\right\}dx = 0$$

where $P_1 = P_1(x)$, $P_2 = P_2(x)$ are as yet arbitrary matrices whose elements are absolutely continuous functions.

We integrate by parts, requiring the matrices P_1, P_2, υ to satisfy the following conditions

$$\begin{aligned}
&\upsilon A + P_1 C = (\upsilon A + P_1 C)^T, \quad P_2 D = (P_2 D)^T \\
&d/dx(P_2 D) = P_2 D_0 + (P_2 D_0)^T, \quad \upsilon B + P_1 D = (P_2 C)^T \\
&d/dx(\upsilon B + P_1 D) = \upsilon B_0 + P_1 D_0 + (P_2 C_0)^T, \quad x \in (0, 1) \\
&[\varphi_T(\upsilon A + P_1 C)\varphi + 2\varphi^T(\upsilon B + P_1 D)\psi + \psi^T P_2 D\psi]_0^1 = 0
\end{aligned} \tag{2.3}$$

Then

$$\frac{dV}{dt} = -\int_0^1 (\varphi^T \omega\varphi - 2\varphi^T \upsilon Q\varphi_\tau)dx \tag{2.4}$$

where

$$\omega = \omega(x) = d/dx(\upsilon A + P_1 C) - (\upsilon A_0 + P_1 C_0) - (\upsilon A_0 + P_1 C_0)^T \tag{2.5}$$

Obviously, as the expression for dV/dt includes the bilinear form $\varphi^T \upsilon Q\varphi_\tau$, the derivative will be a sign-definite form for any values of the vector φ_τ. We will therefore use the stability theorem [2, 3] in which the fact that dV/dt is sign-definite need be verified only for vectors φ_τ in some closed domain.

It follows from the theorem that the solution $\varphi \equiv \psi \equiv 0$ of system (1.1)–(1.3) is stable with respect to the two measures $\rho[\varphi]$ and $\rho_0[\varphi]$ if:

(a) $V[\varphi]$ is continuous with respect to $\rho_0[\varphi]$ at $t = t_0$ and positive definite with respect to $\rho[\varphi]$;

(b) dV/dt (2.4) is non-positive on the set of states such that $V[\varphi(\cdot,\ s)] \leqslant V[\varphi(\cdot,\ t)]$ or $s \in [t - \tau,\ t]$, $t \geqslant t_0$.

Suppose that the elements of the matrix $\upsilon = \upsilon(x)$ are bounded functions. It then follows from the inequality

$$V[(\varphi(\cdot, s)] \leqslant \int_0^1 \lambda_{\max}(x)\varphi^T(x,s)\varphi(x,s)dx \leqslant \sup_{x\in[0,1]} \lambda_{\max}(x)\rho_{01}, \quad s \in [t_0 - \tau, t_0] \tag{2.6}$$

that $V[\varphi]$ is continuous with respect to the measure $\rho_{01}[\varphi]$ (where $\lambda_{\max}(x)$ is the maximum eigenvalue of $\upsilon(x)$ at the point $x \in [0,\ 1]$), Note that for arbitrary functions $\varphi(x,\ s)$ in the space $L_2([0,\ 1] \times [t_0 - \tau,\ t_0])$ the function $V[\varphi]$ is not continuous with respect to the measure $\rho_{02}[\varphi]$ (see the definition of continuity in [3]); in this section, therefore, stability of the trivial solution is considered relative to the measures $\rho[\varphi]$ and $\rho_{01}[\varphi]$.

If the quadratic form $\varphi^T \upsilon(x)\varphi$ is positive definite for all $x \in [0,\ 1]$ and the elements of $\upsilon(x)$ are continuous functions, then $V[\varphi]$ will be positive definite with respect to $\rho[\varphi]$. That this is the case follows from the inequality

$$V \geqslant \int_0^1 \lambda_{\min}(x)\varphi^T(x,t)\varphi(x,t)dx \geqslant \inf_{x\in[0,1]} \lambda_{\min}(x)\rho[\varphi] \tag{2.7}$$

($\lambda_{\min}(x)$ is the minimum eigenvalue of $\upsilon(x)$ at $x \in [0, 1]$).

Thus, if the elements of $\upsilon(x)$ are continuous and bounded functions and the quadratic form $\varphi^T\upsilon(x)\varphi$ is positive definite for all $x \in [0, 1]$), condition (a) of the stability condition will hold.

Condition (b) of that theorem will hold if the corresponding form in the integrand in (2.5) satisfies the inequality

$$\varphi^T\omega(x)\varphi - 2\varphi^T\upsilon(x)Q(x)\varphi_\tau \geqslant 0, \quad x \in [0, 1] \tag{2.8}$$

provided that

$$\varphi_\tau^T\upsilon(x)\varphi_\tau \leqslant \varphi^T\upsilon(x)\varphi, \quad x \in [0, 1] \tag{2.9}$$

Investigating the bilinear form $\varphi^T\upsilon Q\varphi_\tau$ for a conditional extremum as a function of φ_τ, subject to condition (2.9), we obtain the estimate

$$|\varphi^T\upsilon Q\varphi_\tau| \leqslant [\varphi^T\upsilon\varphi \cdot \varphi^T N(x)\varphi)]^{1/2}, \quad N(x) = \upsilon Q\upsilon^{-1}Q^T\upsilon$$

and inequality (2.8) takes the following form when (2.9) holds

$$\varphi^T\omega(x)\varphi - 2[\varphi^T\upsilon(x)\varphi \;\; \varphi^T N(x)\varphi]^{1/2} \geqslant 0, \quad x \in [0,1] \tag{2.10}$$

If the elements of Q are treated as parameters, it follows from (2.10) that the stability domain is defined in the parameter space by the inequality

$$\max_\varphi \frac{\varphi^T\upsilon(x)\varphi \;\; \varphi^T N(x)\varphi}{(\varphi^T\omega(x)\varphi)^2} \leqslant \frac{1}{4}, \quad x \in [0,1] \tag{2.11}$$

The maximum of the function depends only on the direction of the vector φ. This may be verified by expressing φ in the form $\varphi = R\xi$, $\xi^T\xi = 1$, where ξ is the vector of direction cosines and $R = (\varphi^T\varphi)^{1/2}$ is the length of the vector, and then substituting this expression into (2.11).

Further, using the extremal properties of regular pencils of quadratic forms [4]

$$\varphi^T\upsilon\varphi/\varphi^T\omega\varphi \leqslant \lambda$$

we obtain the following estimate (which, though rather rough, is more convenient for practical purposes)

$$\lambda(x)\lambda_\tau(x) \leqslant \frac{1}{4}, \quad x \in [0, 1] \tag{2.12}$$

where $\lambda = \lambda(x)$, $\lambda_\tau = \lambda_\tau(x)$ are the maximum eigenvalues of the matrices $\omega^{-1}(x)\upsilon(x)$ and $\omega^{-1}(x)N(x)$ at the point $x \in [0, 1]$.

Thus, using the Lyapunov function (2.1), we have established sufficient conditions for the trivial solution of system (1.1)–(1.3) to be stable—equalities (2.11) and (2.12).

3. SECOND METHOD OF CONSTRUCTING LYAPUNOV FUNCTIONS

In [3, 5] it was proposed not to use Lyapunov functions, but certain functionals with analogous properties. We will do this here to derive sufficient conditions for the trivial solution of system (1.1)–(1.3) to be stable.

Define a functional V_τ as the sum of two integral quadratic forms

$$V_\tau = \int_0^1 \varphi^T(x,t)\upsilon(x)\varphi(x,t)dx + \int_{t-\tau}^{t}\int_0^1 \varphi^T(x,s)F(x)\varphi(x,s)dxds \tag{3.1}$$

Let us evaluate the derivative dV_τ/dt along trajectories of system (1.1)–(1.3). Proceeding along the same lines as in Section 2, we obtain

$$\frac{dV_\tau}{dt} = -\int_0^1 [\varphi^T(\omega - F)\varphi - 2\varphi^T\upsilon Q\varphi_\tau + \varphi_\tau F\varphi_\tau]dx \tag{3.2}$$

where $\omega = \omega(x)$ is defined by (2.5).

By the method of Lyapunov functions [3], the trivial solution of system (1.1)–(1.3) will be asymptotically stable with respect to the two measures $\rho[\varphi]$ and $\rho_0[\varphi]$ if

(a) V_τ is continuous with respect to the measure $\rho_0[\varphi]$ at $t = t_0$ and positive definite with respect to $\rho[\varphi]$;

(b) dV_τ/dt is negative definite.

Suppose that the elements of the matrix $\upsilon(x)$ and $F(x)$ are bounded functions. Then it follows from the inequality

$$V_\tau \leqslant \sup_{x\in[0,1]} \lambda^\upsilon(x)\rho[\varphi(\cdot,t_0)] + \sup_{x\in[0,1]} \lambda^F(x) \int_{t_0-\tau}^{t_0} \rho[\varphi(\cdot,s)]ds$$

that V_τ is continuous with respect to the measure $\rho_0[\varphi]$, where $\lambda^\upsilon(x)$, $\lambda^F(x)$ are the maximum eigenvalues of υ and F at the point $x \in [0, 1]$. Suppose that the quadratic function $\varphi^T\upsilon(x)\varphi$ in the integrand is positive definite for all $x \in [0, 1]$ and that the elements of $\upsilon(x)$ are continuous functions; suppose, moreover, that the quadratic form $\varphi^T F\varphi$ is non-negative. Then it follows from inequality (2.7) that V_τ is positive definite with respect to $\rho[\varphi]$. We have thus established sufficient conditions for condition (a) to hold.

Condition (b) will be satisfied if the quadratic function of $2n$ variables in the integrand in (3.2) is positive definite for all $x \in [0, 1]$, i.e.

$$V_1[\varphi, \varphi_\tau] = \varphi^T(\omega - F)\varphi - 2\varphi^T\upsilon Q\varphi_\tau + \varphi_\tau^t F\varphi_\tau > 0, \quad x \in [0, 1] \tag{3.3}$$

For this inequality to be true, the following conditions must hold

$$\varphi^T(\omega - F)\varphi > 0, \quad \varphi_\tau^T F\varphi_\tau > 0 \tag{3.4}$$

Define a block matrix

$$M = \| m_{ij} \| = \begin{Vmatrix} \omega - F & -\upsilon Q \\ -(\upsilon Q)^T & F \end{Vmatrix} \quad (i, j = 1, 2, \ldots, 2n)$$

By Sylvester's criterion, a necessary and sufficient condition for the form $V_1[\varphi, \varphi_\tau]$ (3.3) to be positive definite is that the principal minors in the upper left corner of M be positive. This condition requires the evaluation of a large number of determinants. We therefore prefer to use here a recurrent criterion for the positive definiteness of quadratic forms [6].

By this criterion, the quadratic form $V_1[\varphi, \varphi_\tau]$ will be positive definite if and only if

$$m_{ii}(x) - \sum_{k=1}^{i-1} n_{ki}^2(x) > 0 \quad (i = 1, 2, \ldots, 2n), \quad x \in [0, 1] \tag{3.5}$$

where the functions $n_{ij}(x)$ $(i = 1, 2, \ldots, 2n; j \geqslant i)$ are evaluated by the following recurrence relations

$$n_{ii}(x) = \pm\left[m_{ii}(x) - \sum_{k=1}^{i-1} n_{ki}^2(x)\right]^{1/2}$$

$$n_{ij}(x) = \frac{1}{n_{ii}(x)}\left[m_{ij}(x) - \sum_{k=1}^{i-1} n_{ki}(x)n_{kj}(x)\right]; \quad n_{ij}(x) \equiv 0, \quad j < i$$

4. EXAMPLE

Let us investigate the stability of the vibrations of a stretched string in a viscoelastic medium under the influence of a distributed controlling force

$$\frac{\partial^2\varphi(x,t)}{\partial t^2} = \frac{\partial^2\varphi(x,t)}{\partial x^2} - a\frac{\partial\varphi(x,t)}{\partial t} - b\varphi(x,t) + U \tag{4.1}$$

$$x \in (0,1), \quad t \geqslant t_0$$

$$\varphi(0,t) = \varphi(1,t) = 0 \tag{4.2}$$

where $a\partial\varphi(x,t)/\partial t$, $b\varphi(x,t)$ are terms representing the action on the string of dissipative and conservative elastic forces, respectively. We shall assume that $U = -c\varphi(x, t-\tau)$, i.e. the control is achieved by feedback, where τ is the delay of the signal on passing through the feedback loop. Here a and b are the dimensionless drag and coefficient of viscosity of the medium, respectively, and c is the dimensionless amplification factor of the feedback signal.

Equation (4.1) also describes the torsional vibrations of an aircraft, but with different boundary conditions.

Introducing new variables $\varphi_1 = \varphi(x,t), \varphi_2 = \partial\varphi_1\partial t$, $\varphi_3 = \partial\varphi_1/\partial x$ and allowing for the integrability conditions $\partial\varphi_3/\partial t = \partial\varphi_2/\partial x$, (see [7]), we obtain the system

$$\partial\varphi_1/\partial t = \varphi_2, \quad \partial\varphi_2/\partial t = \partial\varphi_3/\partial x - a\varphi_2 - b\varphi_1 - c\varphi_1(x, t-\tau) \tag{4.3}$$

$$\partial\varphi_3/\partial t = \partial\varphi_2/\partial x, \quad \partial\varphi_1/\partial x - \varphi_3 = 0$$

equivalent to Eq. (4.1). Putting

$$A = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{Vmatrix}, \quad A_0 = \begin{Vmatrix} 0 & 1 & 0 \\ -b & -a & 0 \\ 0 & 0 & 0 \end{Vmatrix}, \quad Q = \begin{Vmatrix} 0 & 0 & 0 \\ -c & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}$$

$$C = \|1 \quad 0 \quad 0\|, \quad C_0 = \|0 \quad 0 \quad -1\|$$

we write this system in the form (1.1) and (1.2), where $B = B_0 = D = D_0 = 0$.

To establish sufficient conditions for the solution $\varphi \equiv 0$ of systems (4.1), (4.2) to be stable, we first use the Lyapunov function (2.1), that is

$$V = \gamma\int_0^1 (\upsilon_{11}\varphi_1^2 + 2\upsilon_{12}\varphi_1\varphi_2 + \varphi_2^2 + \varphi_3^2)dx \tag{4.4}$$

where $\upsilon_{11} > 0$, υ_{12}, γ are arbitrary constants. Let us evaluate the derivative dV/dt along trajectories of Eqs (4.2) and (4.3) using the technique described above. For this example, the first and last conditions of (2.3) become

$$\upsilon A + P_1C = (\upsilon A + P_1C)^T, \quad \varphi^T(\upsilon A + P_1C)\varphi|_0^1 = 0 \tag{4.5}$$

If we assume that $P_1 = \|p_1 \quad 0 \quad \upsilon_{12}\|$, $P_2 = \|0\|$, then, taking the boundary conditions (4.2) into consideration, we conclude that conditions (4.5) are satisfied. By (2.5), the matrix ω may be written in the form

$$\omega=\begin{Vmatrix} 2b\upsilon_{12} & b+a\upsilon_{12}-\upsilon_{11} & p_1 \\ b+a\upsilon_{12}-\upsilon_{11} & 2(a-\upsilon_{12}) & 0 \\ p_1 & 0 & 2\upsilon_{12} \end{Vmatrix}$$

To simplify the calculations, we assume that $b\upsilon_{12}=a-\upsilon_{12}$, $\upsilon_{11}=b+a\upsilon_{12}$ $\gamma=(b+1)/(2b)$, $p_1=0$. Then the matrices in (2.1), (2.5) and (2.9) become

$$\upsilon=\frac{b+1}{2b}\begin{Vmatrix} b+a\eta & \eta & 0 \\ \eta & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix},\quad N=\frac{c^2(b+1)}{2b^2(1+\eta^2)}\begin{Vmatrix} \eta^2 & \eta & 0 \\ \eta & 1 & 0 \\ 0 & 0 & 0 \end{Vmatrix}$$

$$\omega=a\operatorname{diag}\{1,\ 1,\ 1/b\},\quad \eta=a/(b+1)$$

Substituting them into (2.11), we obtain the definition of the stability domain in the parameter space (a,b,c)

$$\frac{c^2}{b^2\eta^2(1+\eta^2)}\max_{\varphi\in S}\frac{[(b+a\eta)\varphi_1^2+2\eta\varphi_1\varphi_2+\varphi_2^2+\varphi_3^2](\eta\varphi_1+\varphi_2)^2}{(\varphi_1^2+\varphi_2^2+\varphi_3^2/b)^2}\leqslant 1 \tag{4.6}$$

(S is a sphere of unit radius). For given a and b, the maximum of this function in the compact set S may be determined by numerical optimization methods.

The boundaries of the stability domain in the (a, c) plane as defined by inequality (4.6), are shown in Fig. 1 for $b=3$ and $b=1$ (the solid curves).

We will now determine the stability domain in the (a, c) plane by using the rougher estimate (2.12). this gives

$$\lambda=\frac{1+b+a\eta+[(b+a\eta-1)^2+4\eta^2]^{1/2}}{4b\eta},\quad \lambda_\tau=\frac{c^2}{2\eta b^2}$$

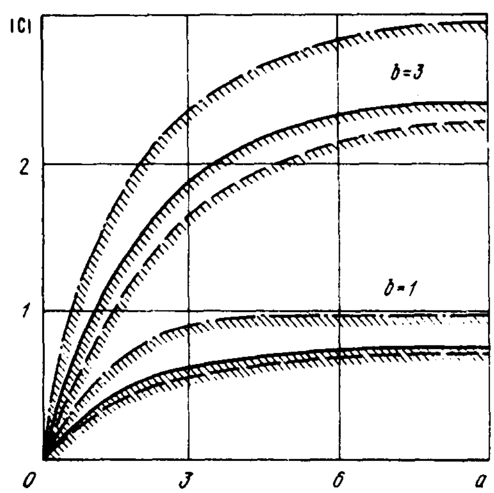

FIG. 1.

Then, by (2.12), the stability domain is defined by the inequality

$$c^2 \leqslant \frac{2b^3\eta^2}{1+b+a\eta+[(b+a\eta-1)^2+4\eta^2]^{1/2}} \tag{4.7}$$

The boundary of the stability domain defined by this inequality is shown in the figure by the dashed curve. Clearly, this inequality gives a somewhat narrower domain. Note, however, that as $a \to \infty$ both inequalities tend to the same limiting values $c_- = b/(b+1)$.

We now use the functional V_τ of (3.1) to determine the asymptotic stability domain with respect to measures $\rho[\varphi]$ and $\rho_0[\varphi]$ in the parameter space (a, b, c). the matrices υ and F are constructed as follows:

$$\upsilon = \frac{1}{2\gamma}\begin{Vmatrix} b+\nu a^2 & \nu a & 0 \\ \nu a & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \quad F = \frac{a}{\gamma}\operatorname{diag}\{f_1, f_2, f_3\}$$

where γ, υ, f_1, f_2, f_3 are as yet arbitrary non-negative constants.

Then, using the recurrent criterion (3.5), we obtain the following inequality defining the stable domain

$$c^2 \leqslant \max_{\substack{0\leqslant\nu\leqslant 1 \\ 0\leqslant f_1\leqslant\nu b}} 4(1-\nu)a^2 \frac{f_1(\nu b - f_1)}{a^2\nu^2(1-\nu)+(\nu b - f_1)} \tag{4.8}$$

Computations were carried out for this case for $b=1$ and $b=3$ and various values of a. the boundary of the stability domain is shown in the figure by the dash–dot curve. This domain is obviously larger than that determined by the previous method.

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Translated by D.L.